CLASSIFICATION OF GROUPS WITH ORDER $\leq$ 20

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BASIC TOOL

Theorem. (Classification of Finite Abelian Groups)

Let G be a finite Abelian group. Then

$$P \cong C_{d_1} \oplus C_{d_2} \oplus \cdots \oplus C_{d_n}.$$

where $d_1, d_2, \dots, d_n$ are (possibly non-distinct) powers of prime numbers, up to reordering.

BASIC TOOL

Theorem. (Sylow Theorem)

Let G be a finite group and p a prime divisor of $|G|$. Then

1. *There exists a p-Sylow subgroup in G.*
2. *If P_1, P_2 are two p-Sylow subgroups in G, then for some g*
$$P_1 = gP_2g^{-1}.$$
3. *Let n be the number of p-Sylow subgroup in G, then*
$$n \equiv 1 \bmod p.$$

BASIC TOOL

Definition. (Semidirect product)

Given a group G, a subgroup H, and a normal subgroup N in G:

$G = N \rtimes H$, *where* $N \cap H = \{e\}$.

The multiplication in G *is given by* $(a_1, b_1) \cdot_G (a_2, b_2) = (a_1 \theta_{b_1}(a_2), b_1 b_2)$.

where $\theta: H \to \mathrm{Aut}(N)$

MAIN IDEA

1. *Locate a normal subgroup in G, call it N.*
2. *Try to find another subgroup H in G that has trivial intersection with N such that $|G| = |N||H|$.*
3. *Then $G = N \rtimes H$. Each possible structure for N, H, and the action of H on N that defines multiplication in G leads to an unique structure for G.*

RESULT

Groups with **Prime Orders $\boldsymbol{p}$**: C_p 1, 2, 3, 5, 7, 11, 13, 17, 19

Groups with **Orders $\boldsymbol{p^2}$**: C_{p^2} $C_p \times C_p$ 4, 9

Act on itself using **left multiplication.**

Use the class formula to prove $Z(G)$ is a **nontrivial $\boldsymbol{p}$-group.**

Use the fact that if $G/Z(G)$ is cyclic then G is Abelian to show **$\boldsymbol{G}$ is Abelian.**

Use the classification theorem.

RESULT

Groups with **Order $\boldsymbol{pq}$**: C_{pq} $C_p \rtimes C_q$ 6, 10, 14

$p < q, q \equiv 1 \bmod p$

Use Sylow theorem to show the ***q-Sylow subgroup is unique*** and thus ***normal***.

$$G \cong \mathrm{Syl}(p) \rtimes \mathrm{Syl}(q)$$

$$\langle a \rangle \qquad \langle b \rangle$$

Action of $\mathrm{Syl}(q)$ on $\mathrm{Syl}(p)$ uniquely determined by ***action of $\boldsymbol{b}$ on $\boldsymbol{a}$***.

Trivial action leads to $\boldsymbol{C_{pq}}$.

Any other action leads to $\boldsymbol{C_p \rtimes C_q}$ after switching generators.

RESULT

Groups with **Order $\boldsymbol{pq}$**: C_{pq} 15

$p < q, q \equiv a \bmod p, a \neq 1$

Use Sylow theorem to show $\boldsymbol{Syl(p)}$ ***and*** $\boldsymbol{Syl(q)}$ ***are normal***.

Use their normality to show $\boldsymbol{Syl(p)}$ ***commutes*** with $\boldsymbol{Syl(q)}$.

$$G \cong \mathrm{Syl}(p) \rtimes \mathrm{Syl}(q)$$

But since the two groups commute, ***the action is trivial***, so the semi product is just a ***direct product***.

RESULT

Groups with **Order 8**: C_8 $C_4 \times C_2$ $C_2 \times C_2 \times C_2$

D_8 Q_8

Use Classification Theorem to handle the Abelian case.

For the non-Abelian case, there must be order 4 element y and another element x not in $\langle y \rangle$.

Conclude the structure based on x^2.

$\boldsymbol{x^2 = e}$ gives $\boldsymbol{D_8}$.

$\boldsymbol{x^2 = y^2}$ gives $\boldsymbol{Q_8}$.

RESULT

Groups with **Order 16**:

C_{16} $\quad C_8 \times C_2$ $\quad C_4 \times C_4$ $\quad C_2 \times C_2 \times C_4$ $\quad C_2 \times C_2 \times C_2 \times C_2$

D_{16} $\quad D_8 \times C_2$ $\quad Q_8 \times C_2$ $\quad C_4 \rtimes C_4$ $\quad Dic_4$ $\quad (C_2 \times C_2) \rtimes C_4$

$\{a, b, c | a^4 = b^2 = c^2 = e, ba = ab, ca = ac, cb = a^2bc\}$

$\{a, b | a^8 = b^2 = e, ba = a^5b\}$

$\{a, b | a^8 = b^2 = e, ba = a^3b\}$

Divides into several cases based on the size and structure of $\boldsymbol{Z(G)}$.

Consider the ***correspondence groups in*** $\boldsymbol{G}$ of ***subgroups in*** $\boldsymbol{G/Z(G)}$.

RESULT

Groups with **Order 12**:	C_{12}	$C_2 \times C_2 \times C_3$	
	D_{12}	A_4	$\langle a, b, c \mid a^3 = b^2 = c^2 = abc \rangle$
Groups with **Order 18**:	C_{18}	$C_3 \times C_3 \times C_2$	
	D_{18}	$S_3 \rtimes Z_2$	E_9
Groups with **Order** 20:	C_{20}	$C_2 \times C_2 \times C_5$	
	D_{20}	$C_5 \rtimes C_4$	$\langle a, b, c \mid a^5 = b^2 = c^2 = abc \rangle$

All three are proved in the same way. I'll explain groups of order 20 in detail.

CLASSIFICATION OF GROUPS OF ORDER 20

$$\begin{tikzcd} & & 20 = 5 \cdot 2^2 \arrow[dl, "\text{UNIQUE}"'] \arrow[dr, "\text{1 OR 5}"] & \\ G = & \mathrm{Syl}(5) & \rtimes & \mathrm{Syl}(2) \end{tikzcd}$$

$$\underbrace{\mathrm{Syl}(5) \rtimes \mathrm{Syl}(2)}_{\text{Trivial Intersection}}$$

We simply need to consider the action θ of $\mathrm{Syl}(2)$ on $\mathrm{Syl}(5)$.

where $\theta : \mathrm{Syl}(2) \to \mathrm{Aut}(\mathrm{Syl}(5))$

Note that $\mathrm{Syl}(5) \cong C_5 = \langle a \rangle$ and $\mathrm{Aut}\big(\mathrm{Syl}(5)\big) \cong C_4$.

CLASSIFICATION OF GROUPS OF ORDER 20

Case I: $\mathrm{Syl}(2) \cong C_4 = \langle b \rangle$.

$\theta_1(b)(a) = a^2$: Let $C_4 \cong \mathrm{Syl}(2)$ identify with C_4 in $\mathrm{Aut}\big(\mathrm{Syl}(5)\big)$.

leads to $C_5 \rtimes C_4$

$\theta_2(b)(a) = a^4$: Send $C_4 \cong \mathrm{Syl}(2)$ to C_2 in $\mathrm{Aut}\big(\mathrm{Syl}(5)\big)$.

Identify $x = (a, b^2), y = (a, b), z = (e, b)$.

leads to $\langle a, b, c | a^5 = b^2 = c^2 = abc \rangle$

$\theta_3(b)(a) = a$: Trivial action.

leads to C_{20}

CLASSIFICATION OF GROUPS OF ORDER 20

Case II: $\mathrm{Syl}(2) \cong C_2 \times C_2 = \{e, b, c, bc\}$.

$\theta_1(b)(a) = a^4$: Let two C_2 in $\mathrm{Syl}(2)$ identify with C_2 in $\mathrm{Aut}\big(\mathrm{Syl}(5)\big)$.
$\theta_1(c)(a) = a^4$ Identify $x = (a, b)$ and $y = (a^3, bc)$.

leads to D_{20}

$\theta_2(b)(a) = a$: Trivial action.
$\theta_2(c)(a) = a$

leads to $C_5 \times C_2 \times C_2$

We have finished our classification.